The original paper follows this 1-page erratum

Erratum to "The Clark-Ocone formula for vector valued random variables in abstract Wiener space", Jour. Func. Anal. 229, 143–154 (2005)

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In this paper we considered the extension of the Clark-Ocone formula for a random variable defined on an abstract Wiener space (W, H, μ) and taking values in a Banach space (denoted there either B or Y). The main result appears in Theorem 3.4. Unfortunately, as first pointed out to us by J. Maas and J. Van Neerven, the dual predictable projection $\mathbf{\Pi}$ introduced in Definition 3.1(iii) via the characterization (3.1), does <u>not</u> define a random operator in $L^2(\mu; L(H, Y))$ as claimed, but rather an element of the larger space $L(H, L^2(\mu, Y))$. Consequently the right hand side of (3.6) in the main result is ill defined.

We have been unable to overcome this difficulty in a meaningful way. We should point out that a Clark-Ocone formula was recently obtained in [3] for random variables on a classical cylindrical Wiener space taking values in a UMD Banach space, in which δ can be explicitly defined à la Itô on adapted processes. Our work, however, was different in spirit and made use of the extended version of δ introduced in [1]. While it is possible to provide an even weaker interpretation of (3.6) in which δ is extended to suitable elements of $L(H, L^2(\mu; Y))$, the result would have amounted to little more than the collection of classical Clark-Ocone formulae for the scalar random variables $\{\langle v, y^*\rangle,\ y^* \in Y^*\}$.

The main result, Theorem 3.4, is thus considerably weakened; it remains true a) assuming that Y^{**} has the Radon Nikodym property (RNP) with respect to μ, and b) for Y-valued random variables v for which one can verify that $\mathbf{\Pi}\nabla v \in L(H, L^2(\mu; Y))$. (The need for the additional RNP condition a) derives from an error, also brought to our attention by J. Maas and J. Van Neerven, in the proof of Proposition 3.14 of [1], (cited here as Lemma 2.3) which has been corrected in [2] under the RNP condition).

Section 4 is not affected by the difficulties described above.

References

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The Clark–Ocone formula for vector valued random variables in abstract Wiener space

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Abstract

The classical representation of random variables as the Itô integral of nonanticipative integrands is extended to include Banach space valued random variables on an abstract Wiener space equipped with a filtration induced by a resolution of the identity on the Cameron–Martin space. The Itô integral is replaced in this case by an extension of the divergence to random operators, and the operators involved in the representation are adapted with respect to this filtration in a suitably defined sense.

Key Words and Phrases: Clark formula, Clark-Ocone formula, Banach space valued random variables, weakly adapted operators,
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1 Introduction

The representation of square integrable functionals of the Wiener process as a sum of multiple Wiener-Itô integrals was derived by K. Itô in his 1951 paper [4]. It follows easily from this series that every such functional is representable as a Itoô integral. This representation, however, was not stated explicitly in [4], and its first appearance seems to have occurred in the 1967 paper of H. Kunita and S. Watanabe [7].

The problem of finding an explicit expression for the integrand in the Itô integral was formulated and solved under certain differentiability restrictions by J. M. C. Clark in 1970 [2]. In 1984, D. Ocone [11] applied the Malliavin calculus to relax these restrictions significantly, and then in further generality with I. Karatzas and J. Li [6]. In loose terms, this representation is valid for L^2 (more generally, L^1) random variables φ on Brownian paths $\omega = (\omega_t)_{0\le t\le 1}$, smooth enough that there exists a ("derivative") process $D_t\varphi$ such that

$$\left.\frac{d\varphi\left(\omega+\varepsilon\int_0^{\cdot} h_s\,ds\right)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^1 D_t\varphi\, h_t\,dt$$

in an appropriate sense. The Clark–Ocone formula then states that

$$\varphi = E\varphi + \int_0^1 E\left(D_t\varphi|\,\mathcal{F}_t\right)\,d\omega_t,$$

where $(\mathcal{F}_t)$ is the canonical filtration.

The purpose of this paper is to obtain the Clark representation for random variables taking values in Banach spaces. This will be done in the context of an abstract Wiener space (W, H, μ) whose natural filtration is induced by a resolution of the identity, thus allowing for the notion of adaptedness. Extensions of the Clark–Ocone formula in an abstract Wiener space have already been studied ([19],[16],[12]) from a different point of view, namely, for scalar random variables.

Section 2 is devoted to some basic notions of stochastic analysis in Wiener space, including the gradient and divergence operators, the latter applied to random variables which are not necessarily H-valued, as introduced in [9]. In Section 3 we first summarize the necessary preliminaries concerning resolutions of the identity, their induced filtrations and vector valued random variables adapted with respect to them, based mostly on [16], [17] and [20]. Next we consider the divergence of (weakly adapted) random variables taking values in a Banach space B (which reduces to the Itô integral when B is the Cameron Martin space)

and then apply these results and those of Section 2 to derive the Clark–Ocone formula for those such variables which are regular . This will be illustrated in Section 4 where measure preserving transformations on Wiener space are considered as W-valued random variables. Section 5 contains some concluding remarks.

2 Stochastic analysis preliminaries

An abstract Wiener space (W, H, μ) consists of a separable Banach space W, a separable Hilbert space H densely embedded in W and a zero mean Gaussian measure μ on W's Borel sets under which each $l \in W^*$ is a $\mathrm{N}(0, |l|_{\mathrm{H}}^2)$ random variable, denoted δl. Here W^* was implicitly taken to be a dense subspace of H, as it will be throughout. By density, this extends to a zero mean linear Gaussian random field $\{\delta h,\ h \in H\}$ whose covariance is induced by H's inner product.

Let (η_n) be an independent sequence of $N(0,1)$ random variables on some probability space $(\Omega, \mathcal{F}, P)$, and (e_n) an orthonormal base (ONB) of H. Itô-Nisio's theorem [5] states that $\sum_{n=1}^{\infty} \eta_n e_n$ converges to a W-valued random variable ξ whose distribution is μ, and that if in particular $\Omega = W$ and $\eta_n = \delta e_n$ for each n, then $\xi(w) = w \ \ \mu$ a.s.

For any Banach space Y and $1 \le p \le \infty$ we denote by $L^p(\mu; Y)$ the class of strongly measurable Y-valued random variables v on W such that $\|v\|_{\mathrm{Y}} \in L^p(\mu)$, and

$$\mathcal{S}(Y) = \Big\{ F := \sum_{j=1}^{m} \underbrace{\varphi_j(\delta h_1, \ldots, \delta h_n)}_{\Phi_j} b_j \ \Big| \ m, n \in \mathbb{N},\ \varphi_j \in C_b^\infty(\mathbb{R}^n),\ h_i \in H,\ b_j \in Y \ \Big\}, \tag{2.1}$$

and the gradient of these *simple* Y-valued random variables is defined to be

$$\nabla F = \sum_{j=1}^{m} \nabla \Phi_j \otimes b_j = \sum_{j=1}^{m} \sum_{i=1}^{n} \partial_i \varphi_j(\delta h_1, \ldots, \delta h_n)\, h_i \otimes b_j \quad \in L^\infty(\mu; L(H, Y)). \tag{2.2}$$

Here and throughout $L(X, Y)$ denotes the space of bounded linear operators from a Banach space X to a Banach space Y, equipped with their operator norm (and $L(X) = L(X, X)$). It should be noted that when Y is a separable Hilbert space, the Hilbert–Schmidt norm of ∇F is traditionally used; the operator norm in this case was first considered by G. Peters in [13].

For each $1 \le p < \infty$ define on $\mathcal{S}(Y)$ the norms

$$\|F\|_{p,1} = \left(\|F\|_{L^p(\mu;Y)}^p + \|\nabla F\|_{L^p(\mu;L(H,Y))}^p \right)^{\frac{1}{p}}. \tag{2.3}$$

The Sobolev spaces $\mathbb{D}_{p,1}(Y) \subset L^p(\mu;Y)$ are defined to be $\mathcal{S}(Y)$'s completions according to these norms. By closability, ∇ can be extended to a bounded operator (with a slight abuse of notation) $\nabla : \mathbb{D}_{p,1}(Y) \to L^p(\mu; L(H,Y))$.

The divergence operator on random operators in $L(H,Y)$ is defined by duality. Recall that the trace $\operatorname{tr} \mathbf{T}$ of an operator $\mathbf{T} \in L(H)$, which is defined to be $\sum_{n=1}^{\infty} {}_{W^*}\langle e_i, \mathbf{T} e_i \rangle_{W^{**}}$ if this sum converges and is the same for every ONB (e_n) of H, induces the pairing $\langle\langle \mathbf{K}, \mathbf{D} \rangle\rangle := \operatorname{tr}\left(\mathbf{K}^T \mathbf{D}\right)$, for appropriate $\mathbf{K} \in L(H,Y)$ and $\mathbf{D} \in L(H,Y^*)$. We shall say that $\mathbf{K} \in L^1(\mu; L(H,Y))$ has finite rank if for some $m \in \mathbb{N}$, $\mathbf{K} = \sum_{k=1}^{m} h_j \otimes y_j$ with $u_j \in L^1(\mu; H)$ and $y_j \in Y$, that is, $\mathbf{K}h = \sum_{j=1}^{m} (u_j, h) y_j$.

Definition 2.1 *For $1 \leq p < \infty$ let $\mathbf{dom}_{p,Y}\,\delta$ be the set of all $\mathbf{K} \in L^p(\mu; L(H,Y))$ for which there exists a $\delta\mathbf{K} \in L^p(\mu; Y^{**})$, the* **divergence** *of $\mathbf{K}$, such that for all $F \in \mathcal{S}(Y^*)$.*

$$E\,\langle\langle \mathbf{K}, \nabla F \rangle\rangle = E\, {}_{Y^*}\langle F, \delta\mathbf{K} \rangle_{Y^{**}} \tag{2.4}$$

(Note that the pairing in (2.1) is well defined since ∇F has finite rank). A necessary and sufficient condition for $\mathbf{K} \in \mathbf{dom}_{p,Y}\,\delta$ (cf. [9, Equation (3.12)]) is that for some $\gamma > 0$

$$|E\,\langle\langle \mathbf{K}, \nabla F \rangle\rangle| \leq \gamma \|F\|_{L^q(\mu;Y^*)}$$

$(\frac{1}{p} + \frac{1}{q} = 1)$ for all $F \in \mathcal{S}(Y^*)$.

Lemma 2.3 below provides a "weak" characterization of $\delta\mathbf{K}$. If δ had been required to be Y–valued (and not only Y^{**}–valued), the "if" implication in the Lemma would no longer be valid.

We denote $\mathbf{dom}_{p,\mathbb{R}}\,\delta = \operatorname{dom}_p \delta$; this space contains H-valued random variables, and in this case δ is the usual divergence.

Remarks 2.2

i) [9, Remark 3.13] If $\mathbf{K}$'s range is μ-a.s. contained in a (deterministic) finite dimensional subspace of Y, (2.4) extends to all $F \in \mathbb{D}_{p,1}(Y^*)$.

ii) If $\alpha \in \operatorname{dom}_p \delta$ and $y \in Y$, it follows directly from the definitions that $\alpha \otimes y \in \mathbf{dom}_{p,Y}\,\delta$ and that $\delta(\alpha \otimes y) = (\delta\alpha) y$.

Lemma 2.3 *[9, Proposition 3.14] An element $\mathbf{K} \in L^p(\mu; L(W^*, Y))$ belongs to $\mathbf{dom}_{p,Y}\,\delta$ if and only if $\mathbf{K}^T l \in \operatorname{dom}_p \delta$ for every $l \in Y^*$ and for some $C > 0$*

$$\|\delta\left(\mathbf{K}^T l\right)\|_{L^p(\mu)} \leq C \|l\|_{Y^*} \qquad \forall\ l \in Y^*. \tag{2.5}$$

In this case

$$\delta(\mathbf{K}^T l) = {}_{Y^*}\langle l, \delta\mathbf{K}\rangle_{Y^{**}} \qquad \text{a.s.} \tag{2.6}$$

and more generally, for any $F \in \mathcal{S}(Y^*)$, $\quad \mathbf{K}^T F \in \mathrm{dom}_p\delta \quad$ *and*

$$\delta(\mathbf{K}^T F) = {}_{Y^*}\langle F, \delta\mathbf{K}\rangle_{Y^{**}} - \langle\!\langle \mathbf{K}, \nabla^{W^*} F\rangle\!\rangle. \tag{2.7}$$

Examples

i) If $v(w) \equiv w_0 \in W$ belongs to $\mathrm{dom}_1\delta$, then necessarily $w_0 \in H$ [9, Remark 3.2b)].

ii) $v(w) = w$ does not belong to $\mathrm{dom}_1\delta$. This follows by applying [9, Proposition 3.6)] to the Itô-Nisio representation $v = \sum_n \delta e_n \, e_n$ for any ONB (e_n).

iii) $v(w) = \sum_{n=1}^{\infty}(\delta e_{2n}\, e_{2n-1} - \delta e_{2n-1}\, e_{2n})$ converges and, like in (ii), $v \overset{\mathcal{D}}{\sim} \mu$ (by the Itô-Nisio theorem). However, here $v \in \mathrm{dom}_1\delta$ and $\delta v = 0$. This follows from [9, Lemmas 3.3, 3.4].

iv) $\mathbf{1}_H$ belongs to $\mathbf{dom}_{p,W}\,\delta$ for all $p \geq 1$ (but not to $\mathbf{dom}_{p,H}\,\delta$!) and $\delta\mathbf{1}_H(w) = w$ μ-a.s. [9, Corollary 3.16)].

3 Adaptedness and the divergence representation of vector-valued random variables

Let $\pi = \{\pi_\theta, \, \theta \in [0,1]\}$ be a strictly increasing continuous resolution of the identity on H (the π_θ's are orthogonal projections in H with $\pi_0 = 0$ and $\pi_1 = I_H$). Each such resolution of the identity induces the filtration $\mathcal{F} = \{\mathcal{F}_\theta, \theta \in [0,1]\}$ on (W, H, μ) defined by

$$\mathcal{F}_\theta = \sigma\left(\delta(\pi_\theta h), \; h \in H\right) \qquad \theta \in [0,1]$$

which generates a time structure with respect to which notions of adaptedness can be defined.

a. Adaptedness

Definitions 3.1 *Let Y be an arbitrary Banach space.*

i) An H-valued random variable u is adapted *(to $\mathcal{F}$) if $\left(u, \pi_\theta h\right)$ is $\mathcal{F}_\theta$–measurable for each $h \in H$ and $\theta \in [0,1]$. $\quad$ Set* $L^2_{\mathrm{a}}(\mu; H) = \left\{u \in L^2(\mu; H), \; u \text{ is adapted}\right\}$.

ii) *An $L(H,Y)$–valued random operator G is* weakly adapted *(to $\mathcal{F}$) if $G^T y^*$ is adapted to $\mathcal{F}$ for each $y^*\in Y^*$. Set* $L^2_{\rm wa}\big(\mu;L(H,Y)\big)=\Big\{G\in L^2\left(\mu;L(H,Y)\right),\ G \text{ is weakly adapted}\Big\}$.

iii) $\Pi=$ *the orthogonal projection of $L^2(\mu;H)$ onto $L^2_{\rm a}(\mu;H)$ and*

$\mathbf{\Pi}\colon L^2\left(\mu;L(H,Y)\right)\longrightarrow L^2_{\rm wa}\big(\mu;L(H,Y)\big)$ *is defined by*

$$ {}_{Y^*}\langle y^*,(\mathbf{\Pi K})h\rangle_{Y}=\Big(\Pi(\mathbf{K}^Ty^*),h\Big)_H, \qquad \mathbf{K}\in L^2\left(\mu;L(H,Y)\right),\ h\in H,\ y^*\in Y^*. \tag{3.1}$$

It follows directly from (3.1) that

$$\Pi\big(\mathbf{K}^Ty^*\big)=(\mathbf{\Pi K})^Ty^* \qquad \forall\mathbf{K}\in L^2\left(\mu;L(H,Y)\right), y^*\in Y^* \tag{3.2}$$

from which it follows that $\mathbf{\Pi K}$ is indeed weakly adapted for every $\mathbf{K}\in L^2\big(\mu;L(H,Y)\big)$. $\mathbf{\Pi}$ is a *projection* onto $L^2_{\rm wa}\big(\mu;L(H,Y)\big)$, as can be easily verified, which moreover inherits from Π the weak orthogonality property

$$E\langle\!\langle\mathbf{K},\mathbf{Q}\rangle\!\rangle=E\langle\!\langle\mathbf{\Pi K},\mathbf{Q}\rangle\!\rangle \tag{3.3}$$

for every $\mathbf{K}\in L^2(\mu;L(H,Y)$ and finite rank $\mathbf{Q}\in L^2_{\rm wa}\big(\mu;L(H,Y^*)\big)$. Indeed, if $Q=q\otimes y^*$, with $q\in L^2_a(\mu;H)$ and $y^*\in Y^*$, then

$$E\operatorname{tr}\mathbf{K}^T(q\otimes y^*)=E\operatorname{tr}q\otimes\mathbf{K}^Ty^*=E(q,K^Ty^*)=E(q,\Pi K^Ty^*),$$

since q is adapted, and the same expression is obtained when $\mathbf{K}$ is replaced by $\Pi\mathbf{K}$.

The following lemma suitably generalizes the Itô integral of adapted processes, and its isometry property

$$E\delta(u)\delta(v)=E(u,v) \qquad \forall u,v\in L^2_a\left(\mu;L(H,Y)\right). \tag{3.4}$$

A random operator $G(\omega):X\to Y$ has finite rank if $G=\sum_{j=1}^m x_j^*\otimes y_j$ for appropriate $m\in\mathbb{N}$, X^*-valued random variables $x_j^*(\omega)$ and nonrandom $y_j\in Y$, $1\le j\le m$.

Lemma 3.2

i) *For any Banach space Y, $L^2_{\rm wa}\big(\mu;L(H,Y)\big)\subset\mathbf{dom}_{2,Y}\,\delta$. If, moreover, $\mathbf{D}\in L^2_{\rm wa}\big(\mu;L(H,Y)\big)$ has finite rank, then $\delta\mathbf{D}\in Y$.*

ii) Given a Banach space B, if $\mathbf{K}\in L^2_{\mathrm{wa}}\big(\mu;L(H,B)\big)$ and $\mathbf{D}\in L^2_{\mathrm{wa}}\big(\mu;L(H,B^*)\big)$ has finite rank, then

$$E\,{}_{B*}\langle\delta\mathbf{D},\delta\mathbf{K}\rangle_{B^{**}}=E\langle\!\langle\mathbf{K},\mathbf{D}\rangle\!\rangle. \tag{3.5}$$

Proof: For any $G\in L^2_{\mathrm{wa}}\big(\mu;L(H,Y)\big)$ and $y^*\in Y^*$, it holds by definition that $G^Ty^*\in L^2_{\mathrm{a}}(\mu;H)$. It is well known that adapted H–valued random variables of second order are Itô integrable, and thus in $\mathrm{dom}_2\delta$. Lemma 2.3 then implies that $G\in\mathbf{dom}_{2,Y}\,\delta$.
If $\mathbf{D}=\sum_{j=1}^m\varphi_j\otimes y_j$, with $\varphi_j\in L^2_{\mathrm{a}}(\mu,H)$ and $y_j\in Y$, $1\le j\le m$, then by Remark 2.2 ii) $\mathbf{D}\in\mathbf{dom}_{2,Y}\,\delta$ and $\delta\mathbf{D}=\sum_{j=1}^m(\delta\varphi_j)y_j$.

As for ii), let $\mathbf{D}=\sum_{j=1}^m u_j\otimes b_j^*$, with $u_j\in L_{\mathrm{a}}(\mu;H)$ and $b_j^*\in B^*$, $1\le j\le m$, and let $(e_i)_{i\in\mathbb{N}}$ be an arbitrary ONB in H. Then

$$\begin{aligned}
E\,{}_{B*}\langle\delta\mathbf{D},\delta\mathbf{K}\rangle_{B^{**}} &= E\sum_{j=1}^m\delta u_j\,{}_{B^*}\langle b_j^*,\delta\mathbf{K}\rangle_{B^{**}}\\
&\overset{(2.6)}{=} \sum_{j=1}^m E\delta(u_j)\,\delta(\mathbf{K}^Tb_j^*)\\
&\overset{(3.4)}{=} \sum_{j=1}^m E\left(u_j,\mathbf{K}^Tb_j^*\right)=E\sum_{j=1}^m\sum_{i=1}^\infty\left(u_j,e_i\right)\left(e_i,\mathbf{K}^Tb_j^*\right)\\
&= E\sum_{j=1}^m\sum_{i=1}^\infty\left(u_j,e_i\right)\,{}_B\langle\mathbf{K}e_i,b_j^*\rangle_{B^*}=E\sum_{i=1}^\infty{}_B\left\langle\mathbf{K}e_i,\sum_{j=1}^m\left(u_j,e_i\right)b_j^*\right\rangle_{B^*}\\
&= E\sum_{i=1}^\infty{}_B\langle\mathbf{K}e_i,\mathbf{D}e_i\rangle_{B^*}=E\langle\!\langle\mathbf{K},\mathbf{D}\rangle\!\rangle. \qquad\square
\end{aligned}$$

Corollary 3.3 *If $\mathbf{K}\in L^2_{\mathrm{wa}}\big(\mu;L(H,Y)\big)$ and $\delta\mathbf{K}=0$ then $\mathbf{K}=\mathbf{0}$.*

Proof: Under the assumptions on $\mathbf{K}$ it follows from (3.5) that $E\langle\!\langle\mathbf{K},\mathbf{D}\rangle\!\rangle=0$ for every finite range weakly adapted random operator $\mathbf{D}:H\to B^*$, in particular $\mathbf{D}=\varphi\otimes b^*$ with $\varphi\in L^2_{\mathrm{a}}(\mu;H)$ and $b^*\in B^*$. Thus

$$0=E\langle\!\langle\mathbf{K},\mathbf{D}\rangle\!\rangle=E\left(\varphi,\mathbf{K}^Tb^*\right),$$

and since φ, b^* were arbitrary, the conclusion follows. $\square$

b. The Clark–Ocone formula

This subsection is devoted to the main result of this note.

Theorem 3.4 *Given a Banach space B and $v \in \mathbb{D}^H_{2,1}(B)$,*

$$v = Ev + \delta\left(\mathbf{\Pi}\nabla v\right) \tag{3.6}$$

and $\mathbf{K} = \mathbf{\Pi}\nabla v$ is the unique element in $L^2_{\mathrm{wa}}\big(\mu; L(H,B)\big)$ such that $v = Ev + \delta\mathbf{K}$.

(By Lemma 3.2i), $\mathbf{\Pi}\nabla v$ indeed belongs to δ's domain.)

Proof: We shall again assume that $Ev=0$. Let $F = \sum_{i=1}^n \Phi_i b_i^* \in \mathcal{S}(B^*)$ be a simple random variable (c.f. (2.1)) for which $E\Phi_i = 0$ for each i. By the standard Itô representation, $\Phi_i = \delta q_i$, for appropriate $q_i \in L^2_{\mathrm{a}}(\mu; H)$, $i = 1, \ldots, n$, so that

$$F = \sum_{i=1}^m \delta(q_i)\, b_i^* = \delta(\mathbf{Q}) \qquad \text{with} \qquad \mathbf{Q} = \sum_{i=1}^m q_i \otimes b_i^* \; \in L^2_{\mathrm{wa}}\big(\mu; L(H, B^*)\big)\,.$$

We shall show that

$${}_B\langle v, F\rangle_{B^*} = E\, {}_B\langle \delta\left(\mathbf{\Pi}\nabla v\right), F\rangle_{B^*} \tag{3.7}$$

from which (3.6) will follow since these test variables F are dense in $L^2(\mu; B^*)$. We have

$$\begin{aligned} E\, {}_B\langle v, F\rangle_{B^*} &= E\, {}_B\langle v, \delta\mathbf{Q}\rangle_{B^*} \\ &= E\langle\!\langle \nabla v, \mathbf{Q}\rangle\!\rangle \\ &= E\langle\!\langle \mathbf{\Pi}\nabla v, \mathbf{Q}\rangle\!\rangle \\ &= E\, {}_B\langle \delta\left(\mathbf{\Pi}\nabla v\right), \delta\mathbf{Q}\rangle_{B^*} = E\, {}_B\langle \delta\left(\mathbf{\Pi}\nabla v\right), F\rangle_{B^*}, \end{aligned}$$

where Remark 2.2 i) was used in the second equality, (3.3) in the third and Lemma 3.2 ii) in the fourth.

As for the uniqueness, if $v = \delta\mathbf{K}_i$ with $\mathbf{K}_i \in L^2_{\mathrm{wa}}\big(\mu; L(H,B)\big)$, $i = 1, 2$, it follows that $\delta\left(\mathbf{K}_1 - \mathbf{K}_2\right) = 0$ and thus $\mathbf{K}_1 = \mathbf{K}_2$ by Corollary 3.3. $\square$

4 Measure preserving transformations on the Wiener space

Let (W, H, μ) be an abstract Wiener space and let $e_i, i = 1, 2, \ldots$ take values in W^* and such that the images of the e_i in H are a complete orthonormal base on H. By the Ito-Nisio

theorem [5]

$$w_n = \sum_1^n \delta(e_i)\, e_i \tag{4.1}$$

with e_i considered as elements in W, converges in L_1 on W to w, similarly if $\{\eta_i, i = 1, 2, \dots\}$ are i.i.d., $N(0,1)$ then $\sum_1^n \eta_i e_i$ converges in L_1 on W to a W-valued random variable which has the same probability law as w. In this case $Tw := \sum_1^\infty \eta_i e_i$ will be denoted an "abstract Wiener process" or "a measure preserving transformation on the Wiener space" or (for reasons that will become clear later) "a rotation". Note that w and Tw, while each being Gaussian are, in general, not jointly Gaussian. The fact that Tw as defined above is W-valued suggests the problem of the Clark representation of this transformation. We have already noted that for $Tw = w, w = \delta(I)$. The analysis and characterization of measure preserving transformations is not new ([18],[20]) and most of the results presented here are known; it is, however, more natural to analyze the class of measure preserving transformations in the context of this section.

We prepare the following result for later reference:

Proposition 4.1 *Let $\mathbf{R}(w)$ be an a.s. bounded operator on H. Assume that $\mathbf{R}(w)$ is weakly adapted with respect to a filtration induced by a continuous increasing π. Since $\mathbf{R}h$ is adapted it is in the domain of δ. Assume that the probability law of $\delta(\mathbf{R}h)$ is $N(0, |h|_H^2)$, then:*

1. *If $h_1, h_2 \in H$ and $(h_1, h_2)_H = 0$ then $\delta(\mathbf{R}h_1)$ and $\delta(\mathbf{R}h_2)$ are independent.*
2. *$\mathbf{R}(w)$ is a.s. an isometry on H.*
3. *$\sum_i \delta(\mathbf{R}e_i)\, e_i$ is measure preserving, and if (e_i) and (h_i) are ONB's of H then, a.s.,*

$$\sum_i \delta(\mathbf{R}h_i) h_i = \sum_i \delta(\mathbf{R}e_i)\, e_i \,. \tag{4.2}$$

Proof:

1. $$\begin{aligned} E\,\exp\{i\alpha\delta(\mathbf{R}h_1)\}\exp\{i\beta\delta(\mathbf{R}h_2)\} &= E\exp\Big\{\delta\Big(\alpha h_1+\beta h_2\Big)\Big)\Big\} \\ &= E\exp\left\{-\frac{\alpha^2}{2}\,|h_1|_H^2-\frac{\beta^2}{2}\,|h_2|_H^2\right\} \\ &= E\exp\{i\alpha\delta(\mathbf{R}h_1)\}\,E\exp\{i\beta\delta(\mathbf{R}h_2)\}\,. \end{aligned}$$

2. By part 1, $y_\theta=\delta(\mathbf{R}\pi_\theta h)$ is a Gaussian process of independent increments. Hence it is Gaussian martingale and its quadratic variation satisfies

$$\langle y,y\rangle_\theta = Ey_\theta^2\,. \tag{4.3}$$

and by our assumption $Ey_\theta^2=|\pi_\theta h|_H^2$. But

$$\langle y,y\rangle_\theta = (\mathbf{R}\pi_\theta h,\mathbf{R}\pi_\theta h)_H \tag{4.4}$$

and $\mathbf{R}^T\mathbf{R}=I$ follows.

3. Follows from the Ito-Nisio theorem. □

Theorem 4.2 *Let (W,H,μ) be an abstract Wiener space and let $\{\pi_\theta,\ \theta\in[0,1]\}$ be a strictly increasing continuous resolution of the identity on H, and $\mathcal{F}$ its induced filtration. If Tw is a measure invariant transformation on (W,H,μ) then there exists a $\mathbf{R}\in L^2_{\mathrm{wa}}\big(\mu;L(H,W)\big)$ which is a.s. an isometry on H, such that*

$$Tw=\delta\mathbf{R}\,. \tag{4.5}$$

Conversely if $\mathbf{R}\in L^2_{\mathrm{wa}}\big(\mu;L(H,W)\big)$ is a.s. an isometry on H then $\mathbf{R}\in\mathbf{dom}_{2,W}\,\delta$ and $\delta\mathbf{R}$ is measure preserving.

(Note that almost surely R's range is contained in H, but its divergence is W–valued).

Proof: By our assumptions, every η_i can be uniquely represented as $\eta_i=\delta u_i$ where the u_i are adapted, in the domain of δ, and $u_i\in\mathbb{D}_2(H)$. Define $\mathbf{R}$ by

$$\mathbf{R}(w)e_i=u_i \tag{4.6}$$

then $\mathbf{R}(w)$ is weakly adapted, and satisfies the assumptions of the previous result. Hence $\mathbf{R}$ is an isometry and $Tw=\sum\delta(\mathbf{R}e_i)e_i$. In the converse direction, since $\mathbf{R}(w)$ is weakly adapted, by Corollary 2.6.1 of [18], $m_\theta=\delta(\pi_\theta\mathbf{R}h)$ $\theta\in[0,1]$ is a $\mathcal{F}_\theta$ square integrable

martingale and $\langle m\rangle_\theta = |\pi_\theta \mathbf{R}h|_H^2$. Consequently by the Girsanov (or the stronger Novikov) condition

$$\begin{aligned} 1 &= E\ \exp\left\{\delta(\mathbf{R}h) - \frac{1}{2}|\mathbf{R}h|\right\} \\ &= E\ \exp\left\{\delta(\mathbf{R}h) - \frac{1}{2}|h|^2\right\} . \end{aligned}$$

It follows that $\delta(\mathbf{R}h)$ is $N(0,|h|^2)$ and that $\delta(\mathbf{R}e_i)$ are i.i.d. $N(0,1)$, so that

$$Tw = \sum \delta(\mathbf{R}e_i)\, e_i = \delta\mathbf{R}\, . \qquad \square$$

5 Concluding Remarks

There is certainly no uniqueness in the representation of a random variable as a divergence if adaptedness of the integrand is not required. If a scalar random variable ϕ, for example, can be written as $\phi = \delta v$, and if

$$U_0 = \{u \in \mathrm{dom}_2\delta,\ \ \delta u = 0\}$$

(that is, U_0 is the nonempty class of "divergence free" integrands), then $\phi = \delta(v+u)$ for any $u \in U_0$. The same is true for vector valued random variables.

The question arises if there is a canonical integrand $\bar{v}$, for example

$$E\|\bar{v}\|_H^2 = \min\left\{E|v|_H^2,\ \phi = \delta v\right\} \tag{5.1}$$

or equivalently

$$E(v,u)_H = 0 \quad \forall u \in U_0 \qquad \text{(i.e. } v \in U_0^\perp) \ .$$

If we denote $L_\mathrm{e}^2(\mu;H) := \{\nabla F,\ F \in \mathbb{D}_{2,1}\}$ the space of *exact* H–valued random variables, then clearly $L_\mathrm{e}^2(\mu;H) \subset U_0^\perp$ since $E(\nabla F, u) = EF\delta u$. Thus if $\phi = \delta(\nabla F)$ for some $\nabla F \in L_\mathrm{e}^2(\mu;H)$, then $\bar{v} = \nabla F$ is the (necessarily unique) integrand which satisfies (5.1).

Let $\mathcal{L} = \sum_{n=0}^\infty n\mathcal{P}_n$ be the Ornstein–Uhlenbeck, or number, operator on $L^2(\mu)$, where $\mathcal{P}_n$ is $L^2(\mu)$'s projection onto its nth homogeneous chaos, and $\mathrm{dom}\mathcal{L}$ is the appropriate domain of convergence. From its definition, we see that $\mathcal{L}$'s restriction to $\mathrm{dom}\mathcal{L} \cap \{\phi \in L^2(\mu),\ E\phi = 0\}$ has a bounded inverse. In addition, it is well known that $\phi \in \mathrm{dom}\mathcal{L}$ if and only if $\phi \in \mathbb{D}_{2,1}$ and $\nabla\phi \in \mathrm{dom}\delta$, in which case $\mathcal{L}\phi = \delta\nabla\phi$.

From the above discussion we conclude that

$$\phi = E\phi + \delta\left(\nabla\mathcal{L}^{-1}(\phi - E\phi)\right), \tag{5.2}$$

and that $\bar{v} = \nabla\mathcal{L}^{-1}(\phi - E\phi)$ is the unique exact integrand in terms of which ϕ can be represented as a divergence, and as such satisfies the minimality condition (5.1). Note that $\bar{v}$ is in general quite different from the adapted integrand discussed in this work; they coincide if and only if ϕ belongs to the first chaos $\mathcal{P}_1(L^2(\mu))$.

The Ornstein–Uhlenbeck operator $\mathcal{L}$ can be defined just as well in $L^2(\mu; B)$ for any Banach space B (cf. for example [14]) via its interpretation as the generator of the Ornstein–Uhlenbeck semigroup. However, in order to extend (5.2) to B–valued ϕ's, assumptions on B seem to be needed in this case to conclude that $\mathcal{L}$ has a bounded inverse on $L^2(\mu; B)$'s subspace of zero expectation, and this restricts the extension of the above argument when trying to obtain (5.2) for vector valued random variables.

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